# Su(2) intelligent states as coupled su(2) coherent states

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We show how su(2) intelligent states can be obtained by coupling su(2) coherent states. The construction is simple and efficient, and easily leads to a discussion of some general properties of su(2) intelligent states.

## I. INTRODUCTION

In quantum mechanics, uncertainty relations give a lower bound on the uncertainty resulting from the simultaneous measurement of two non–commuting observables. One common uncertainty relation was obtained in [1]: if $\hat{\Omega}$ and $\hat{\Lambda}$ are self-adjoint operators, and if $|\psi\rangle$ is a state normalized to 1, then we have

$$\Delta\Omega\Delta\Lambda \geq \tfrac{1}{2}\,|\langle[\hat{\Omega},\hat{\Lambda}]\rangle|. \tag{1}$$

In Eq.(1), $\Delta\Omega$ is the standard deviation of the operator $\hat{\Omega}$ for a quantum system described by $|\psi\rangle$ , *i.e.*

$$\Delta\Omega = \sqrt{\langle\hat{\Omega}^2\rangle - \langle\,\hat{\Omega}\,\rangle^2}, \tag{2}$$

with $\langle\,\hat{X}\rangle = \langle\psi|\,\hat{X}\,|\psi\rangle$.

In this paper, we will discuss su(2) states for which the strict equality in Eq.(1) holds, *i.e.* su(2) states for which ($\hbar = 1$)

$$\Delta L_x \Delta L_y = \tfrac{1}{2}|\langle\hat{L}_z\rangle|. \tag{3}$$

States that satisfy Eq.(3) are known as su(2) intelligent states. The terminology was first introduced by Aragone *et al* [2]. It is clear that the right hand side of Eq.(3) depends on the choice of state used to evaluate $\langle\hat{L}_z\rangle$, so intelligent states need to be distinguished from minimum uncertainty states; there are intelligent states for which the rhs of Eq.(3) is not the obvious minimum value of 0.

By su(2) state, we understand a (pure) quantum state $|\psi\rangle$ that belongs to an irreducible representation of the su(2) algebra. This algebra is spanned by the familiar angular momentum operators $\{\hat{L}_x, \hat{L}_y, \hat{L}_z\}$ or, more conveniently, by the complex linear combinations $\{\hat{L}_\pm, \hat{L}_z\}$, where

$$\begin{aligned} \hat{L}_\pm &= \hat{L}_x \pm i\hat{L}_y\,, \\ \left[\hat{L}_z, \hat{L}_\pm\right] &= \pm\,\hat{L}_\pm\,, \qquad \left[\hat{L}_+, \hat{L}_-\right] = 2\hat{L}_z\,. \end{aligned} \tag{4}$$

An irreducible representation of dimension $2j+1$, where $j$ can be an integer or a half–integer, is spanned by the set $\{|jm\rangle, m = -j, -j+1, \ldots, j-1, j\}$ with

$$\hat{L}_z|jm\rangle = m\,|jm\rangle\,, \qquad \hat{L}_\pm|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j, m\pm 1\rangle\,. \tag{5}$$

Intelligence is not limited to su(2) states. A well–known example of intelligent states is the harmonic oscillator coherent state $|\,\xi\rangle$, parameterized by the complex number $\xi$ and for which

$$\Delta x \Delta p = \tfrac{1}{2}\,. \tag{6}$$

However, in this paper, we understand intelligent states to mean su(2) intelligent states.

The terminology "su(2) intelligent states" is to be contrasted with recent theoretical and experimental work [3][4][5] on angular momentum states of light as quantum states carrying orbital angular momentum about the beam axis. In these papers, the spectrum of the operator $\hat{L}_z$ is unbounded, leading to a differential eigenvalue equation rather than the finite–dimensional eigenvalue problem of Eq.(12).

An important ingredient to our construction will be the su(2) coherent states [6]. It is sufficient here to recall the well–known property that such states are obtained by a rotation of the extremal su(2) state $|\ell, \ell\rangle$. More specifically, an su(2) coherent state $|\gamma, \vartheta\rangle$ can be parameterized by two angles $\gamma, \vartheta$ such that, up to an overall phase

$$|\gamma, \vartheta\rangle = R_z(\gamma) R_y(\vartheta) R_z(-\gamma)|\ell\,, \ell\rangle\,, \tag{7}$$

where $R_i(\varphi)$ denotes the rotation about the axis $i$ by an angle $\varphi$. Su(2) coherent states with $\gamma = 0$ or $\pi/2$ also satisfy Eq.(3) . However, su(2) intelligent states are not always of the form of Eq.(7).

Indeed, we plan to show that all intelligent states are of the form

$$[R_y(\beta)|\ell_A\,, \ell_A\rangle] \otimes [R_y(-\beta)|\ell_B\,, \ell_B\rangle]\ , \tag{8}$$

$$\text{or} \qquad [R_x(\beta)|\ell_A\,, \ell_A\rangle] \otimes [R_x(-\beta)|\ell_B\,, \ell_B\rangle]\ , \tag{9}$$

corresponding to Eqn.(7) with $\gamma = 0$ or $\pi/2$ and a specific choice of $\vartheta$.

Su(2) intelligent states of angular momentum $\ell$ are of the form

$$\begin{aligned} &\hat{\Pi}^{\ell}\ [R_y(\beta)|\ell_A\,,\ell_A\rangle] \otimes [R_y(-\beta)|\ell_B\,,\ell_B\rangle]\ , \\ \text{or}\quad &\hat{\Pi}^{\ell}\ [R_x(\beta)|\ell_A\,,\ell_A\rangle] \otimes [R_x(-\beta)|\ell_B\,,\ell_B\rangle]\ , \end{aligned} \tag{10}$$

with $\ell = \ell_A + \ell_B$ and where $\hat{\Pi}^{\ell} = \sum_m |\ell, m\rangle\langle\ell, m|$ is the (non–unitary) operator that projects into the $\ell$ subspace.

Thus, our work functions as a bridge between the work of Hillery and Mlodinow [7] and the work of Rashid[8]. In [7], some intelligent states were obtained as su(2) coherent states. They correspond to setting $\ell_B = 0$ in Eqn.(10). No projection is required and, although not every su(2) intelligent state can be constructed, the use of a single unitary transformation means that these states are amenable to experimental implementation [9]. The construction method of [8] is distinctive in that it requires the use of a non–unitary transformation, although it completely solves the construction problem in a single shot.

Eqn.(10), on the other hand, lends itself to a clear physical interpretation: to construct a general intelligent state of angular momentum $\ell$, we must bring together two separate systems, each of which has been subjected to a different unitary transformation, and then extract from this combined system states of good angular momentum using a non–unitary operation akin to a measurement of $\ell$. This interpretation provides a much clearer picture of su(2) intelligent states than the one presented in [10].

In addition to [7] and [8], the original work [2] of Aragone *et al.* has blossomed in various directions. In particular, the recent work of [11] deals with entanglement and su(2) intelligent states. Generalized intelligent states, which satisfy

$$\Delta\Omega^2\Delta\Lambda^2 = \tfrac{1}{4}\langle[\hat{\Omega},\hat{\Lambda}]\rangle^2 + \tfrac{1}{4}\langle\{\hat{\Omega}-\langle\Omega\rangle\,,\hat{\Lambda}-\langle\Lambda\rangle\}\rangle^2, \tag{11}$$

where $\{\hat{\Omega},\hat{\Lambda}\} \equiv \hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}$, have been the object of considerable attention (see, for instance, [12]), including various applications in quantum optics [13][14][15]. Several authors, in particular [16], have studied spin squeezing using the construction of [8]. Trifonov [17] has studied multi–observables and multidimensional generalizations of Eq.(11).

Our work is organized as follows. We first identify a simple but basic property of solutions of the eigenvalue problem; this is encapsulated in Eq.(24). Once this is done, the eigenvalue problem associated with intelligence is solved explicitly for spin-$\frac{1}{2}$ in Sec.III. These spin-$\frac{1}{2}$

states and Eq.(24) are used in Sec.IV A to construct, using a minimum amount of extra work, all intelligent states of angular momentum $\ell = {}^5\!/_2$. This method is generalized to arbitrary $\ell$ in Sec.IV B. The general expression for our angular momentum state can be found in Eq.(60). Some simple analytical and numerical results are presented in Sec.V. A discussion and a short conclusion can be found in Sec.VI.

## II. SOME SIMPLE PROPERTIES

Recall [18] that intelligent states $\left|\psi^\ell\left(\alpha\right)\right\rangle$ of angular momentum $\ell$ are eigenstates of the non-hermitian operator $\hat{L}_x - i\alpha\hat{L}_y$, *i.e.* they satisfy

$$(\hat{L}_x - i\alpha\hat{L}_y)|\psi^\ell(\alpha)\rangle = \lambda\,|\psi^\ell(\alpha)\rangle, \tag{12}$$

where $-\infty \leq \alpha \leq \infty$ is a real parameter. The eigenvalue $\lambda$ is related to the average value of $\hat{L}_x$ and $\hat{L}_y$ and to the parameter $\alpha$ via:

$$\lambda = \langle\,\hat{L}_x\rangle - i\alpha\langle\,\hat{L}_y\rangle\,. \tag{13}$$

Equation (12) stems from two requirements. To replace the inequality of Eq.(1) by the equality and obtain Eq.(11), the states $(\hat{L}_x - \langle\hat{L}_x\rangle)\,|\psi^\ell(\alpha)\rangle$ and $(\hat{L}_y - \langle\hat{L}_y\rangle)\,|\psi^\ell(\alpha)\rangle$ must be collinear, *i.e.*

$$(\hat{L}_x - \langle\hat{L}_x\rangle)\,|\psi^\ell(\alpha)\rangle = i\alpha(\hat{L}_y - \langle\hat{L}_y\rangle)\,|\psi^\ell(\alpha)\rangle. \tag{14}$$

We obtain intelligence by forcing the anticommutator term in Eq.(11) to 0:

$$\langle\psi^\ell(\alpha)|\{\hat{L}_x - \langle\hat{L}_x\rangle, \hat{L}_y - \langle\hat{L}_y\rangle\}|\psi^\ell(\alpha)\rangle = 0. \tag{15}$$

This restricts the values of $\alpha$ to be real and produces Eq.(12).

Let us now abstractly consider a composite system made from two independent subsystems, denoted by the subscripts $A$ and $B$ respectively, such that

$$\hat{L}_{x,A} \equiv \hat{L}_x \otimes \mathbb{1}_B\,, \quad \hat{L}_{x,B} \equiv \mathbb{1}_A \otimes \hat{L}_x\,, \tag{16}$$

$$\hat{L}_x = \hat{L}_{x,A} + \hat{L}_{x,B}\,, \tag{17}$$

where $\mathbb{1}_A$ and $\mathbb{1}_B$ are unit operators in their respective subspaces. Eq.(16) simply means that $\hat{L}_{x,A}$ acts on the first (or “A”) subsystem only, leaving the second (or “B”) subsystem

alone, and similarly for $\hat{L}_{x,B}$. The operators

$$\hat{L}_y = \hat{L}_{y,A} + \hat{L}_{y,B}, \tag{18}$$

$$\hat{L}_z = \hat{L}_{z,A} + \hat{L}_{z,B} \tag{19}$$

are defined in a similar obvious manner.

Let $|\chi(\alpha)\rangle_A$ and $|\phi(\alpha)\rangle_B$ be states of subsystems A and B, respectively, with the property that

$$(\hat{L}_{x,A} - i\alpha\hat{L}_{y,A})|\chi(\alpha)\rangle_A = \lambda_A|\chi(\alpha)\rangle_A \tag{20}$$

$$(\hat{L}_{x,B} - i\alpha\hat{L}_{y,B})|\phi(\alpha)\rangle_B = \nu_B|\phi(\alpha)\rangle_B\,, \tag{21}$$

*i.e.* $|\chi(\alpha)\rangle_A$ and $|\phi(\alpha)\rangle_B$ are intelligent in their respective subsystems. Then,

$$|\psi(\alpha)\rangle = |\chi(\alpha)\rangle_A \otimes |\phi(\alpha)\rangle_B \equiv |\chi(\alpha)\rangle_A |\phi(\alpha)\rangle_B \tag{22}$$

is intelligent since

$$\begin{aligned}(\hat{L}_x - i\alpha\hat{L}_y)|\psi(\alpha)\rangle &= \left[(\hat{L}_{x,A} - i\alpha\hat{L}_{y,A})|\chi(\alpha)\rangle_A\right]|\phi(\alpha)\rangle_B \\ &\quad +|\chi(\alpha)\rangle_A\left[(\hat{L}_{x,B} - i\alpha\hat{L}_{y,B})|\phi(\alpha)\rangle_B\right], \qquad (23) \\ &= (\lambda_A + \nu_B)|\chi(\alpha)\rangle_A|\phi(\alpha)\rangle_B. \qquad (24)\end{aligned}$$

In other words, the direct product of two intelligent states is also intelligent, provided that one thinks of the resulting state as a composite state constructed from two separate systems. This simple result is quite powerful as it indicates that intelligent states can be "built-up" by putting together other intelligent states.

Quite clearly, the task now at hand is to find the simplest intelligent states and use them as building blocks to construct more complicated ones.

## III. INTELLIGENT STATES WITH $\ell = 1/2$

Consider the simplest realization of $\hat{L}_x - i\alpha\hat{L}_y$. Using basis states $|+\rangle$ and $|-\rangle$, for which

$$\hat{L}_z \mapsto \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{L}_x \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{L}_y \mapsto \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{25}$$

we obtain the $2 \times 2$ matrix

$$\hat{L}_x - i\alpha \hat{L}_y \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1-\alpha \\ 1+\alpha & 0 \end{pmatrix}. \tag{26}$$

The (unnormalized) eigenstates, which are by definition intelligent states, are just

$$\begin{pmatrix} 1 \\ \frac{1+\alpha}{\sqrt{1-\alpha^2}} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1+\alpha}{\sqrt{1-\alpha^2}} \end{pmatrix}, \tag{27}$$

with respective eigenvalues

$$\lambda_+ = \lambda \equiv \tfrac{1}{2}\sqrt{1-\alpha^2}\,, \qquad \lambda_- = -\lambda\,. \tag{28}$$

Introducing the quantity

$$\mu = \frac{1+\alpha}{\sqrt{1-\alpha^2}}, \tag{29}$$

we obtain the normalized intelligent states as

$$|\psi_-^{1/2}(\mu)\rangle = \frac{1}{\sqrt{1+|\mu|^2}}\begin{pmatrix} 1 \\ -\mu \end{pmatrix} = \frac{1}{\sqrt{1+|\mu|^2}}|+\rangle - \frac{\mu}{\sqrt{1+|\mu|^2}}|-\rangle, \tag{30}$$

$$|\psi_+^{1/2}(\mu)\rangle = \frac{1}{\sqrt{1+|\mu|^2}}\begin{pmatrix} 1 \\ \mu \end{pmatrix} = \frac{1}{\sqrt{1+|\mu|^2}}|+\rangle + \frac{\mu}{\sqrt{1+|\mu|^2}}|-\rangle. \tag{31}$$

We note that, if $|\alpha| < 1$, $\mu$ is real and we can write

$$|\psi_\pm^{1/2}(\beta)\rangle = R_y(\pm\beta)|+\rangle = \mathrm{e}^{\mp i\beta\,\hat{L}_y}|+\rangle\,, \tag{32}$$

with

$$\cos\tfrac{\beta}{2} = \tfrac{1}{\sqrt{1+|\mu|^2}}\,, \quad \sin\tfrac{\beta}{2} = \tfrac{\mu}{\sqrt{1+|\mu|^2}}\,. \tag{33}$$

*From Eqn.(7), we see that the spin-$1/2$ intelligent states are also coherent states when $\mu$ is real.*

On the other hand, when $|\alpha| \geq 1, \mu$ is purely imaginary and we have

$$|\psi_\pm^{1/2}(\beta)\rangle = R_x(\pm\beta)|+\rangle = \mathrm{e}^{\mp i\beta\hat{L}_x}|+\rangle\,, \tag{34}$$

where, this time,

$$\cos\tfrac{\beta}{2} = \tfrac{1}{\sqrt{1+|\mu|^2}}\,, \quad i\sin\tfrac{\beta}{2} = \tfrac{\mu}{\sqrt{1+|\mu|^2}}\,. \tag{35}$$

*From Eqn.(7), we see that the spin-$1/2$ intelligent states are also coherent states when $\mu$ is purely imaginary.*

## IV. GENERAL CONSTRUCTION

### A. Example: An intelligent states with $\ell = {}^5\!/_2$

We can use the states $|\psi_\pm^{1/2}(\beta)\rangle$ of Eqns.(30) and (31) to construct $\ell = {}^5\!/_2$ intelligent states as follows. Consider the product

$$\begin{aligned} |\psi_{+++--}(\beta)\rangle \;=\; & \Big[|\psi_+^{1/2}(\beta)\rangle_1\,|\psi_+^{1/2}(\beta)\rangle_2\,|\psi_+^{1/2}(\beta)\rangle_3\Big] \\ & \otimes \Big[|\psi_+^{1/2}(\beta)\rangle_4\,|\psi_+^{1/2}(\beta)\rangle_5\Big] \ . \end{aligned} \tag{36}$$

Here, the index $i$ labels one of five spin-$\frac{1}{2}$ subsystems. If we expand every $|\psi_+^{1/2}(\beta)\rangle_i$ and distribute the product, the first term of the resulting expression is given by

$$|\ell = {}^5\!/_2\,, m = {}^5\!/_2\rangle = \cos^5\tfrac{\beta}{2}\,(|+\rangle_1|+\rangle_2|+\rangle_3|+\rangle_4|+\rangle_5) \ . \tag{37}$$

This term is fully symmetric under permutation.

Let us use the shorthands

$$\begin{aligned} \hat{L}_{x,1} &= \hat{L}_x \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \mathbb{1}_5 \ , \\ \hat{L}_{x,2} &= \mathbb{1}_1 \otimes \hat{L}_x \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \mathbb{1}_5 \end{aligned} \tag{38}$$

*etc.*, so that each $\hat{L}_{x,i}$ acts only on the $i$'th subspace (of dimension 2). Let

$$\begin{aligned} \hat{L}_{x,A} &= \hat{L}_{x,1} + \hat{L}_{x,2} + \hat{L}_{x,3} \ , \\ \hat{L}_{x,B} &= \hat{L}_{x,4} + \hat{L}_{x,5} \ , \end{aligned} \tag{39}$$

and define

$$\hat{L}_x = \hat{L}_{x,A} + \hat{L}_{x,B} \ . \tag{40}$$

The collective operators $\hat{L}_y$ and $\hat{L}_z$ are defined similarly, as are $\hat{L}_\pm$:

$$\hat{L}_\pm = \hat{L}_x \pm i\,\hat{L}_y \ . \tag{41}$$

Because the collective operators are fully symmetric under permutation of any two subspace index $i$ in Eq.(40), and act on the symmetric state $|\frac{5}{2},\frac{5}{2}\rangle$, every state of angular momentum $\ell = {}^5\!/_2$ will be symmetric under permutation. Thus, the order in which the $|+\rangle$'s or $|-\rangle$'s occur is unimportant.

### 1. *The case* $|\alpha| < 1$

With $|\alpha| < 1$, every $|\psi_{\pm}^{1/2}(\beta)\rangle$ is obtained by rotation about the $y$–axis. Thus, we can write

$$|\psi_{+++--}(\beta)\rangle = \left[R_y^A(\beta)|\tfrac{3}{2},\tfrac{3}{2}\rangle_A\right]\left[R_y^B(-\beta)|1,1\rangle_B\right] , \tag{42}$$

where we have directly coupled

$$\begin{aligned}[R_y(\beta)|+\rangle_1]\otimes[R_y(\beta)|+\rangle_2]\otimes[R_y(\beta)|+\rangle_3] &= R_y^A(\beta)\left[|+\rangle_1|+\rangle_2|+\rangle_3\right] , \\ &= R_y^A(\beta)|\tfrac{3}{2},\tfrac{3}{2}\rangle_A , \end{aligned}\tag{43}$$

$$[R_y(-\beta)|+\rangle_4]\otimes[R_y(-\beta)|+\rangle_5] = R_y^B(-\beta)|1,1\rangle_B , \tag{44}$$

Here, the rotation operator $R_y^A(\beta) = \mathrm{e}^{-i\beta\hat{L}_{y,A}}$ while $R_y^B(-\beta) = \mathrm{e}^{i\beta\hat{L}_{y,B}}$. Note that the states of Eqs.(43) and (44) are both angular momentum coherent states.

Eq.(42) can now be expanded as

$$\sum_{m_A,m_B} |\tfrac{3}{2},m_A\rangle_A\,|1,m_B\rangle_B\,d^{3/2}_{m_A,3/2}(\beta)\,d^1_{m_B,1}(-\beta) , \tag{45}$$

where

$$d^\ell_{m,m'}(\beta) \equiv \langle\ell,m|\,R_y(\beta)|\ell,m'\rangle \tag{46}$$

is the reduced Wigner function [19].

To project into the $\ell = 5/2$ subspace, we specialize the projector

$$\hat{\Pi}^\ell = \sum_{m=-\ell}^{\ell} |\ell,m\rangle\langle\ell,m| \tag{47}$$

to $\ell = 5/2$ so as to obtain

$$|\psi^{5/2}_{+++--}(\beta)\rangle \propto \sum_m |5/2,m\rangle\,\kappa^{5/2,m}_{3/2,1}(\beta) , \tag{48}$$

where

$$\kappa^{\ell m}_{\ell_A,\ell_B}(\beta) = \sum_{m_A(m_B)} C^{\ell m}_{\ell_A,m_A;\ell_B,m_B}\times d^{\ell_A}_{m_A,\ell_A}(-\beta)\,d^{\ell_B}_{m_B,\ell_B}(\beta) , \tag{49}$$

and $C^{\ell m}_{\ell_A,m_A;\ell_B,m_B}$ is an su(2) Clebsch-Gordan coefficient.

A better, more compact notation for $|\psi^{5/2}_{+++--}\rangle$ is

$$|\psi^{5/2}_{+++--}\rangle \equiv |\psi^{5/2}_{3/2,1}(\beta)\rangle . \tag{50}$$

This emphasizes that only the total number of $|+\rangle_i$ states and the total number of $|-\rangle_j$ states are relevant for the construction of an intelligent state of angular momentum $\ell = \ell_A + \ell_B$. The state $|\psi^{5/2}_{++-+-}(\beta)\rangle$, for instance, can differ from $|\psi^{5/2}_{+++--}(\beta)\rangle$ by at most a phase.

To show that the state of Eq.(50) is intelligent, we note that the operator $\hat{\Pi}^{5/2}$ of Eq.(47) acts as the unit operator on any state completely in the $\ell = 5/2$ subspace, and annihilates any state with no part in this subspace. Hence, the collective $\hat{L}_y = \hat{L}_{y,A} + \hat{L}_{y,B}$ operator and its $\hat{L}_x$ counterpart must commute with the projection $\hat{\Pi}^{5/2}$ of Eq.(47) since neither $\hat{L}_y$ nor $\hat{L}_x$ can change $\ell$. Thus,

$$\left(\hat{L}_y - i\alpha\hat{L}_x\right)|\psi^{5/2}_{3/2,1}(\beta)\rangle = \hat{\Pi}^{5/2}\left(\hat{L}_y - i\alpha\hat{L}_x\right)|\psi_{3/2,1}(\beta)\rangle\,, \tag{51}$$

$$= (3\lambda_+ + 2\lambda_-)|\psi^{5/2}_{3/2,1}(\beta)\rangle\,. \tag{52}$$

The projection does not preserve the norm so $|\psi^{5/2}_{3/2,1}(\beta)\rangle$ must be normalized after the projection.

Since $R^A_y(\beta)|\frac{3}{2},\frac{3}{2}\rangle_A$ and $R^B_y(-\beta)|1,1\rangle_B$ are coherent, we see that $|\psi^{5/2}_{3/2,1}(\beta)\rangle$ is the result of coupling two su(2) coherent states.

#### *2. The case $|\alpha| \geq 1$.*

In this case, we note that

$$\begin{aligned}&\langle \ell, m|\, R_x(\beta)|\ell,\ell\rangle \\ &\quad = \langle \ell, m|\, R_z(-\pi/2)\, R_y(\beta)\, R_z(\pi/2)\,|\ell,\ell\rangle\,, \qquad (53)\\ &\quad = \mathrm{e}^{-i\pi(\ell-m)/2} d^\ell_{m,\ell}(\beta)\,, \qquad (54)\end{aligned}$$

so that, for instance,

$$|\psi^{5/2}_{3/2,1}(\beta)\rangle \propto \sum_m |\tfrac{5}{2}, m\rangle\, \mathrm{e}^{-i\pi(\frac{5}{2}-m)/2}\, \kappa^{5/2 m}_{3/2,1}(\beta)\,, \tag{55}$$

is intelligent by the same argument given for the $|\alpha| < 1$ case.

### B. A general expression

More generally, it is now clear that if we start with $2\ell_A$ copies of $|\psi^{1/2}_+(\beta)\rangle$ and $2\ell_B$ copies of $|\psi^{1/2}_-(\beta)\rangle$, we can write

$$\left[R^A_y(\beta)|\ell_A,\ell_A\rangle\right] \otimes \left[R^B_y(-\beta)|\ell_B,\ell_B\rangle\right]\,, \tag{56}$$

and project into a good $\ell$ subspace using Eq.(47) to obtain an intelligent state of angular momentum $\ell = \ell_A + \ell_B$ as

$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle \propto \sum_m |\ell, m\rangle\, \kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)\,, \tag{57}$$

with $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$ given in Eq.(49).

Eqs.(56) and (57) show explicitly how su(2) intelligent states with angular momentum $\ell$ can be constructed by appropriately coupling su(2) coherent states. The state of Eq.(56) is explicitly intelligent and remains intelligent under projection by $\hat{\Pi}^{\ell}$ of Eq.(47), thus yielding Eq.(57).

We show in A how $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$ can be reduced to

$$\kappa^{\ell,m}_{\ell_A\ell_B}(\beta) = 2^{\ell}\frac{\sqrt{(2\ell_B)!\,(2\ell_A)!\,(\ell+m)!\,(\ell-m)!}}{(2\ell)!}\, d^{\ell}_{\ell_B-\ell_A,m}\left(\tfrac{\pi}{2}\right)\, d^{\ell}_{m,\ell}(\beta). \tag{58}$$

Introducing the norm

$$\mathcal{N}^{\ell}_{\ell_A,\ell_B}(\beta) = \frac{1}{\sqrt{\sum_m |\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)|^2}}\,, \tag{59}$$

we obtain the final expression for our intelligent state as

$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle = \mathcal{N}^{\ell}_{\ell_A,\ell_B}(\beta)\,\sum_m |\ell, m\rangle\, \kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)\,. \tag{60}$$

Finally, we note that the eigenvalue problem in the $\ell = \ell_A + \ell_B$ subspace has at most $2\ell+1$ independent eigenvectors. Using Eq.(60), it is clear that (except when $\beta = 0$ or $\pi$) we can construct exactly the right number linearly independent states of the form by selecting in turn $(\ell_A, \ell_B)$ to be $(\ell, 0), (\ell - 1/2, 1/2), \ldots, (0, \ell)$. Hence, all $2\ell + 1$ intelligent states are coupled su(2) coherent states.

When $\beta = 0$ or $\pi$, $\alpha$ is $\mp 1$, the operator $\hat{L}_x - i\alpha\hat{L}_y$ is nilpotent and has a single eigenvector: either $|\ell, \ell\rangle$ or $|\ell, -\ell\rangle$. This is (indirectly) illustrated in Figures 1 and 2, where it is that all uncertainty curves merge to a single curve at $\beta = \pi$ (or $\alpha = 1$).

## V. SELECTED RESULTS

### A. Expectations and standard deviations

The intelligent state of Eq.(60) is an eigenstate of $\hat{L}_x - i\alpha\hat{L}_y$ with eigenvalue

$$\lambda_{\ell_A,\ell_B} = \lambda\,(2\ell_A - 2\ell_B)\,. \tag{61}$$

If we assume $|\alpha| \leq 1$, then $\lambda$ is real. Combining Eqs.(28),(29) and (33), we obtain

$$\lambda_{\ell_A,\ell_B} = (\ell_A - \ell_B)\sin\beta\,. \tag{62}$$

Since $\alpha$, $\langle\, \hat{L}_x \rangle$ and $\langle\, \hat{L}_y \rangle$ are real, this can be compared with $\lambda_{\ell_A,\ell_B} = \langle\, \hat{L}_x \rangle - i\alpha \langle\, \hat{L}_y \rangle$ to give

$$\langle\, \hat{L}_x \rangle = \tfrac{1}{2}\,(\ell_B - \ell_A)\sin\beta\,, \qquad \langle\, \hat{L}_y \rangle = 0\,. \tag{63}$$

If, on the other hand, $|\alpha| \geq 1$, we have

$$\langle\, \hat{L}_y \rangle = -\tfrac{1}{2}\,(\ell_B - \ell_A)\sin\beta\,, \qquad \langle\, \hat{L}_x \rangle = 0\,. \tag{64}$$

Furthermore, using Eqs.(14) and (15), one finds that the intelligent states generally satisfy

$$(\Delta L_y)^2 = -\frac{1}{2\alpha}\,\langle\, \hat{L}_z \rangle\ , \qquad (\Delta L_x)^2 = -\frac{1}{2}\alpha\,\langle\, \hat{L}_z \rangle\,. \tag{65}$$

This allows computation of all pertinent quantities from $\langle\, \hat{L}_z \rangle$, which is simply given by

$$\langle\, \hat{L}_z \rangle = \left(\mathcal{N}^{\ell}_{\ell_A\ell_B}(\beta)\right)^2 \left(\sum_m m\,|\,\kappa^{\ell m}_{\ell_A,\ell_B}(\beta)\,|^2\right)\,. \tag{66}$$

### B. Numerical results

Figures 1 and 2 illustrate typical results. The figures give the ratio of the uncertainty products $(\Delta L_x\, \Delta L_y)_I$ of intelligent states to the coherent state $(\Delta L_x\, \Delta L_y)_c$, for which $\ell_A = \ell$. These ratios are just the ratios of $\langle \hat{L}_z \rangle$. For the coherent state, one rapidly finds

$$\langle\, \hat{L}_z\, \rangle_c = \frac{\ell}{2}\cos\beta\,, \tag{67}$$

for $|\alpha| < 1$.

In figure 1, the ratios for intelligent states of angular momentum $\ell = {}^5\!/_2$ with $(\ell_A = 2, \ell_B = {}^1\!/_2)$ and $(\ell_A = {}^3\!/_2, \ell_B = 1)$ are given. The results are unchanged if one switches $\ell_A$ and $\ell_B$. The curves $\alpha < 0$ are identical to those for $\alpha > 0$. Furthermore, the results with $|\alpha| > 1$ can be obtained from those with $|\alpha| < 1$ by the transformation $\alpha \to -{}^1\!/_\alpha$, so the range $0 \leq \alpha \leq 1$ captures all qualitative features of the curves. Figure 2 is similar to 1, except that $\ell = 3$. The symmetries of Fig.1 are also present in Fig. 2.

One immediately observes that the uncertainty products for intelligent states (with $\ell_A \neq \ell$) is always greater than the corresponding product for the coherent state (with $\ell_A = \ell$).

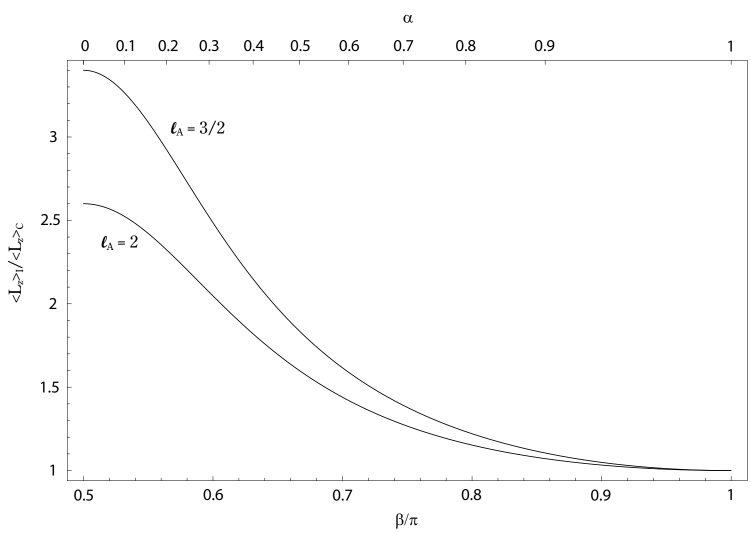


FIG. 1: The ratio $|\langle \hat{L}_z \rangle|_I / |\langle \hat{L}_z \rangle|_c$ as a function of $^{\beta}/_{\pi}$ or $\alpha$ for $\ell = {}^5/_2$ and various values of $\ell_A$ and $\ell_B$ so that $\ell_A + \ell_B = {}^5/_2$.

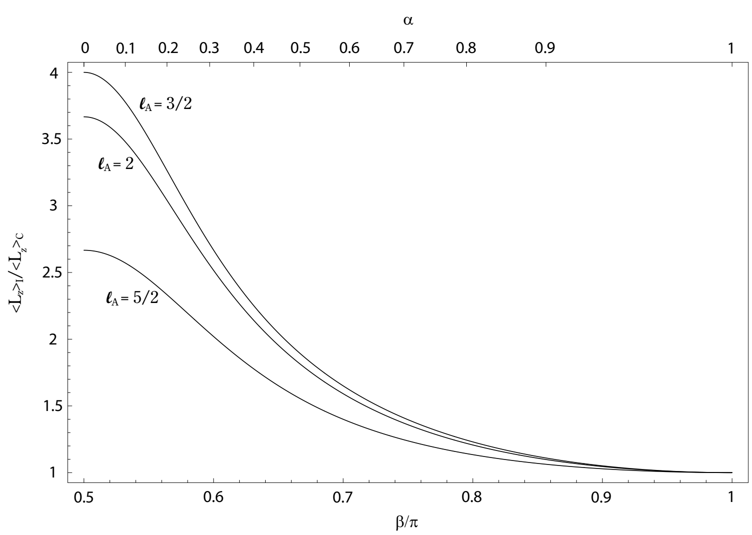


FIG. 2: The ratio $|\langle \hat{L}_z \rangle|_I / |\langle \hat{L}_z \rangle|_c$ as a function of $^{\beta}/_{\pi}$ or $\alpha$ for $\ell = 3$ and various values of $\ell_A$ and $\ell_B$ so that $\ell_A + \ell_B = 3$.

Insofar as the product $\Delta L_x \, \Delta L_y$ goes, the "worst" intelligent state is the one for which $\ell_A$ and $\ell_B$ are as close as possible. We have not been able to prove this analytically because the expression (66) for $\langle \hat{L}_z \rangle$ is difficult to manipulate. However, we have verified that this observation holds over a wide range of values of $\ell$. Other curves illustrating this behavior

can be found in [10].

It is not difficult to show that the maximum of the product $\Delta L_x \Delta L_y$ is simply $\frac{1}{2}\ell$. Indeed, by Eq.(3), it is clear that the product is maximal when $|\langle \hat{L}_z \rangle|$ is maximal. This maximum is reached for the states $|\ell, \pm\ell\rangle$. From Eqs.(56) and (57), it immediately follows that this will occur when $\beta = 0$ or $\beta = \pi$. This, implies by Eq.(33) that $\mu = 0$ or $\mu = \infty$ which in turn,by Eq.(29), implies $\alpha = \pm 1$.

As $\alpha \to \pm 1$, all intelligent states converge to a single state. When $\alpha = \pm 1$ precisely, the operator $\hat{L}_x - i\alpha \hat{L}_y$ becomes the nilpotent $\hat{L}_+$ or $\hat{L}_-$ respectively, both of which have only one non-zero eigenvector.

Figure 3 shows the population of various $m$ substates in the intelligent state $|\psi^{5/2}_{3/2,1}(\beta)\rangle$. For clarity, we have restricted the calculations to angles $\beta$ chosen so that $\langle \hat{L}_z \rangle = \pm 3/2, \pm 1/2$.

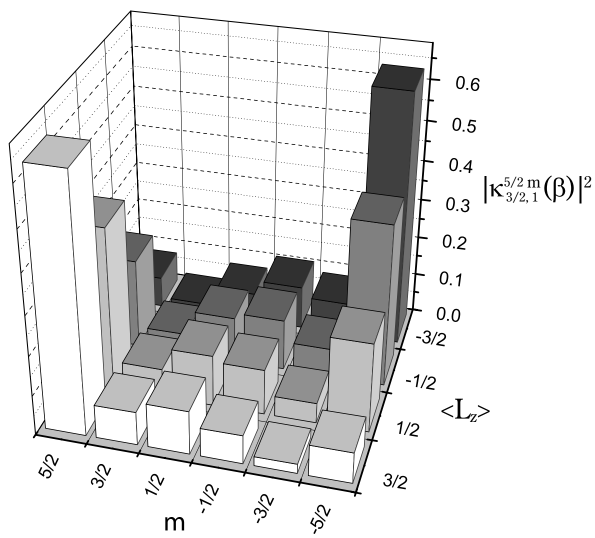


FIG. 3: The populations of $m$ substates $|\kappa^{5/2\,m}_{3/2,1}(\beta)|^2$ for different values of $m$ and $\ell = 5/2$. The values of $\beta$ were selected so that $\langle \hat{L}_z \rangle = \pm 3/2, \pm 1/2$.

This figure illustrates a very general symmetry: $|\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)|^2 = |\kappa^{\ell,-m}_{\ell_A,\ell_B}(-\beta)|^2$. This can be traced back to symmetries of the $d$-functions entering in the construction of the $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$ coefficients.

## VI. DISCUSSION AND CONCLUSION

Let us construct the operators

$$\hat{K}_i = \hat{L}_{iA} - \hat{L}_{iB}. \tag{68}$$

The operators $\{\hat{K}_x, \hat{K}_y, \hat{K}_z\}$ do not close under commutation. However, $\hat{K}_x, \hat{K}_y, \hat{L}_z$ do close on an angular momentum algebra, which we call the $K$- angular momentum $su(2)_K$. This set is interesting because our intelligent states are constructed using a $K$- rotation about $y$. Indeed, defining $\hat{K}_\pm$ in the usual manner, one can see that the state

$$|\ell_A, \ell_A\rangle|\ell_B, \ell_B\rangle\,, \tag{69}$$

is an eigenstate of $\hat{L}_z$ with eigenvalue $m_K = \ell_A + \ell_B = \ell$. Because (69) is killed by $\hat{K}_+$, it can be identified with the state $|\ell, m_K = \ell\rangle_K$ of $K$-angular momentum. In particular, our starting state

$$\begin{aligned}\left[R_y^A(\beta)\,|\ell_A, \ell_A\rangle\right]\left[R_y^B(-\beta)\,|\ell_B, \ell_B\rangle\right] \;&=\; \exp\left[-i\beta\left(\hat{L}_{yA} - \hat{L}_{yB}\right)\right]|\ell, \ell\rangle_K \\ &=\; \exp\left[-i\beta\hat{K}_y\right]|\ell, \ell\rangle_K \end{aligned} \tag{70}$$

and is thus a $K$-angular momentum coherent state.

Unfortunately, the $K$-angular momenta do not commute with the collective angular momenta $\hat{L}_i$. Although (69) is simultaneously a state with "good" total $\ell, m_\ell$ and "good" $\ell_K = \ell\,, m_K$, other $|\ell, m_K \neq \ell\rangle_K$ states generated by the action of $\hat{K}_-$ do not have "good" $\ell, m_\ell$; hence the need for the projection into the subspace of good $L$- angular momentum.

In [10], a method of constructing all intelligent states of angular momentum $\ell$ was proposed. The basic polynomials $\xi^x$ and $\eta^y$ are related to the direct product of $x$ copies of $|+\rangle$ and the direct product of $y$ copies $|-\rangle$, respectively, via the correspondences

$$|+\rangle \leftrightarrow \xi\,, \quad |-\rangle \leftrightarrow \eta\,, \quad |\ell, m\rangle \leftrightarrow \frac{\xi^{\ell+m}\eta^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}\,. \tag{71}$$

Using this, we can write, for $|\alpha| < 1$, the intelligent state $|\psi^\ell_{\ell_A,\ell_B}(\beta)\rangle$ as the product

$$|\psi^\ell_{\ell_A,\ell_B}(\beta)\rangle = \left(\xi\,\cos\tfrac{\beta}{2} + \eta\,\sin\tfrac{\beta}{2}\right)^{2\ell_A}\,\left(\xi\,\cos\tfrac{\beta}{2} - \eta\,\sin\tfrac{\beta}{2}\right)^{2\ell_B}\,. \tag{72}$$

There is no need for projection as the result is a polynomial of total degree $2\ell = 2(\ell_A + \ell_B)$. It is well–known that the polynomials of the form $\xi^x\eta^y$, with $x + y = 2\ell$, span a basis for the

su(2) representation of angular momentum $\ell$. The combinatorics involved in the expansion of Eq.(72) and the conversion of various $\xi^x\eta^y$ to angular momentum states yield precisely Eq.(60). Thus, we recover in a much more transparent way the construction and calculations of [10]. (An expression similar to Eq.(72) can easily be found for $|\alpha| \geq 1$.)

The simple form of Eqs.(60),(65) and (66) illustrate the economy inherent to an approach based on coupling. These results can be contrasted, for instance, with the corresponding expressions of [8] or the application done by [22] of su(2) intelligent states in nuclear physics.

Our results, which only require a table to Clebsch-Gordan coefficient and expressions for Wigner $D$-function, represent the simplest example of what could be a systematic algorithm for the construction of intelligent states of observables elements of other Lie algebras [20] or even deformed algebras [21]. In other words, the procedure presented here is easily generalizable. Indeed, using the results of [23][24], the properties of some SU(3) intelligent states will be the topic of a forthcoming paper[20].

## VII. ACKNOWLEDGMENTS

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## APPENDIX A: THE COEFFICIENT $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$

The expression of Eq.(58) can be manipulated into a more transparent form using [19]

$$d^{\ell_B}_{m_B,\ell_B}(-\beta) \;=\; (-1)^{m_B-\ell_B}\, d^{\ell_B}_{m_B,\ell_B}(\beta)\,, \tag{A1}$$

$$d^{\ell_A}_{m_A,\ell_A}(\beta)\, d^{\ell_B}_{m_B,\ell_B}(\beta) \;=\; C^{\ell\, m}_{\ell_A,m_A;\ell_B,m_B} \times d^{\ell}_{m,\ell}(\beta)\,, \tag{A2}$$

where $\ell = \ell_A + \ell_B$ and $C^{\ell\,\ell}_{\ell_A,\ell_A;\ell_B,\ell_B} = 1$ have been used. Thus,

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = d^{\ell}_{m,\ell}(\beta) \times \left[ \sum_{m_A(m_B)} (-1)^{m_B-\ell_B} \left(C^{\ell\, m}_{\ell_A,m_A;\ell_B,m_B}\right)^2 \right]\,. \tag{A3}$$

A little more mileage can be done because Clebsch-Gordan coefficients for which $\ell = \ell_A + \ell_B$ have known expressions [19]. Using this and the condition $m = m_A + m_B$, we obtain

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = \frac{d^{\ell}_{m,\ell}(\beta)}{\binom{2\ell}{\ell-m}} \left[ \sum_{n=0}^{2\ell_B} (-1)^n \binom{2\ell_A}{\ell-m-n} \binom{2\ell_B}{n} \right] . \tag{A4}$$

The coefficient in the bracket can be identified with the coefficient of $x^{\ell-m}$ in the expansion of $(1+x)^{2\ell_A}(1-x)^{2\ell_B}$. In particular, when $\ell_A = \ell_B$, $\ell$ is integer and there can be no odd powers of $x$, so that no odd values of $m$ will appear in the expansion.

Finally [19],

$$\sum_{n=0}^{2\ell_B} (-1)^n \binom{2\ell_A}{\ell-m-n} \binom{2\ell_B}{n} = 2^{\ell} \sqrt{\frac{(2\ell_B)!\,(2\ell_A)!}{(\ell+m)!\,(\ell-m)!}}\, d^{\ell}_{\ell_B-\ell_A,m}\left(\tfrac{\pi}{2}\right) . \tag{A5}$$

Inserting this into $\kappa^{\ell,m}_{\ell_A\ell_B}(\beta)$ gives Eq.(58).

Note that the appearance of a rotation by $\pi/2$ about the $\hat{y}$ axis:

$$\begin{aligned} d^{\ell}_{m,\ell_B-\ell_A}\left(\tfrac{\pi}{2}\right) &= d^{\ell}_{\ell_B-\ell_A,m}\left(-\tfrac{\pi}{2}\right) \\ &= \langle \ell, \ell_B-\ell_A | \mathrm{e}^{-i\frac{\pi}{2}\hat{L}_y} | \ell, m\rangle , \end{aligned} \tag{A6}$$

is reminiscent of an expression found in [8].

Lastly, although we have limited ourselves to expressions where $\ell = \ell_A + \ell_B$, the factor $\ell_B - \ell_A$ makes it clear that, up to a normalization, it is only the difference between angular momenta that is here relevant. More precisely, if one considers $\ell'_A = \ell_A + j, \ell'_B = \ell_B + j$, then the tensor product $\ell'_A \otimes \ell'_B$ will contain a subspace of angular momentum $\ell$. The coupled states in this subspace are also intelligent, but are simply proportional to the state obtained by coupling $\ell_A \otimes \ell_B$. In other words, no new state is found by considering cases other than $\ell = \ell_A + \ell_B$.

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